Int J Solids Structures Vol. 23, No 1, pp 1 10, 1987
Printed in Great Britain.

0020 7683/87 $3.00 + .00
Pergamon Journals Ltd.

ELASTIC INTERACTION OF A CRACK WITH A MICROCRACK ARRAY—I. FORMULATION OF THE PROBLEM AND GENERAL FORM OF THE SOLUTION

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(*Received* 2 *November* 1984; *in revised form* 11 *September* 1985)

Abstract—Elastic interactions of a crack with an array of microcracks located near the tip is considered. The analysis is based on the potential representations (known also as representation of cracks by dislocations) and approximation of tractions on the microcracks by polynomials.

INTRODUCTION

Elastic interaction of a crack with a damage field is considered; the latter being modelled as a field of microcracks. Vast literature exists on elastic solutions for solids with cracks. These solutions are either based on the complex variables technique for the 2-D problems or on numerical procedures or obtained as asymptotic estimates for remotely located cracks (see, e.g. extensive reviews[1,2] and recent publications[3–5]). In this paper a specific configuration consisting of a dominant crack and an array of microcracks located in a close vicinity of a crack tip ("small-scale" microcracking) is analyzed. The method of analysis is based on the combination of the double layer potential technique (integral representation known also as representation of a crack by dislocations) and Willis' polynomial conservation theorem[6]. It is applicable to microcrack arrays of an arbitrary geometry. An advantage of such representations is that they are applicable to both 2- and 3-D problems. Also, the unknown functions—CODs, or crack shapes—have simple physical meaning (unlike complex potentials) which suggests the approximating technique. Since the crack–damage interaction rather than the general many cracks problem is the primary objective of this work, its relation to the literature on elastic crack solutions and various series techniques is not discussed.

The displacement field in a linear elastic solid generated by a single crack can be represented by an integral of the double layer potential type, with the displacement discontinuity–crack opening displacement (COD) $\mathbf{b} = \mathbf{b}(\mathbf{x})$ being the potential density[7] as

$$\mathbf{u}(\mathbf{x}) = \int_{\omega} \mathbf{b}(\mathbf{x}')\cdot\mathbf{\Phi}(\mathbf{x}',\mathbf{x})\,d\mathbf{x}' \tag{1}$$

where ω is a crack surface (line l, in a two-dimensional case) and $\mathbf{\Phi}$ is the second Green's tensor which can be interpreted as the displacement response at point $\mathbf{x}$ to a unit intensity force dipole applied at point $\mathbf{x}'$ and the integral is to be understood in the principal value sense when $\mathbf{x} \to \mathbf{x}'$. In the plane stress problem, the second Green's tensor is given as

$$\mathbf{\Phi}(\mathbf{x}',\mathbf{x}) = \frac{(1+\nu)}{4\pi R^2}\left[\frac{1-\nu}{1+\nu}(\mathbf{n}_{x'}\mathbf{R} - \mathbf{R}\mathbf{n}_{x'} - \mathbf{n}_{x'}\cdot\mathbf{R}\mathbf{I}) - 2\frac{\mathbf{n}_{x'}\cdot\mathbf{R}}{R^4}\mathbf{R}\mathbf{R}\right] \tag{2}$$

where $\mathbf{n}_{x'}$ is the unit normal vector in the direction of the force dipole at point x', $\mathbf{I}$ is the isotropic tensor, $\mathbf{R} = \mathbf{x}' - \mathbf{x}$ is the position vector and ν is the Poisson's ratio.

The stress field generated by a crack is given by

$$\sigma(\mathbf{x}) = \mathbf{T}_x \int_{\omega} \mathbf{b}(\mathbf{x}') \cdot \mathbf{\Phi}(\mathbf{x}', \mathbf{x}) \, d\mathbf{x}' \tag{3}$$

where $\mathbf{T}_x$ denotes the stress operator transforming $\mathbf{u}$ into stresses

$$T_{ij}\{\mathbf{u}\} = \mu(u_{i,j} + u_{j,i}) + \lambda u_{kk}\delta_{ij} \tag{4}$$

where μ and λ are Lame's constants, δ_{ij} is Kronecker's delta, and $i, j = 1,2$. Index "x" indicates that differentiation in $\mathbf{T}$ is performed with respect to $\mathbf{x}$. Note that representations (1) and (3) are valid in both two- and three-dimensional cases with a suitable second Green's tensor $\mathbf{\Phi}$ chosen.

Using representations (1) and (3) for the system "crack–microcrack array" the problem can be reduced to the one of finding vectorial functions–microcrack CODs $\mathbf{b}(\mathbf{x})$ and the stress intensity factor K at the macrocrack tip from the system of integral equations expressing the boundary conditions on the crack faces. The microcrack CODs are sought in the form (ellipse) × (polynomial) where the first multiplier corresponds to the crack embedded into a uniform stress field and the second multiplier accounts for crack interactions. Such a representation is based on approximation of the stress field induced along the line of a given crack by other cracks by polynomials and the theorem on polynomial conservation[6] stating that the COD of a crack embedded into a polynomial stress field of degree N has the form (ellipse) × (polynomial of degree N). Thus, the system of singular integral equations reduces to a system of linear algebraic equations for the polynomials' coefficients. This system can be easily solved if the number of microcracks is not too large. If, however, microcracks are numerous, the system becomes inconvenient for analysis. For this case the alternative iterative approach, leading to approximate analytical solution, is suggested. The iterations have clear physical meaning: the zeroth iteration corresponds to the stress field σ^∞ due to remotely applied loads in the absence of cracks, the first iteration gives the stress field generated by non-interacting cracks embedded into the σ^∞-field, the second and next iterations correspond to the first order, double, triple and higher order multiplicity of crack interactions. Note, also, that the iterative approach is particularly convenient if the geometry of the microcrack array is known only in statistical terms.

In Part I, the formulation of the problem and general form of the solution are considered. In Part II, the technique is applied to two simple 2-D configurations (involving one and two microcracks) and the problem of "stress shielding" and "stress amplification" are discussed.

FORMULATION OF THE PROBLEM

The configuration considered here is two-dimensional and consists of the main crack $(-l_0, l_0)$ with the crack tips surrounded by microcrack arrays. The stress field in the vicinity of the macrocrack tip can be represented as a superposition:

$$\sigma(\mathbf{x}) = \sigma^\infty + \hat{\sigma}(\mathbf{x}) + \sum_{i=1}^{N} \sigma_i(\mathbf{x}) \tag{5}$$

where σ^∞ is the stress field due to remotely applied loads in the absence of all cracks; $\hat{\sigma}$ and σ_i are the stress fields generated by the main crack and by the ith microcrack, correspondingly. More exactly, σ_i is the stress field in an infinite elastic plane containing one ith crack, with faces loaded by tractions

$$\mathbf{n}_i \cdot \left[\boldsymbol{\sigma}^{\infty} + \hat{\boldsymbol{\sigma}}(\mathbf{x}_i) + \sum_{\substack{k=1 \\ k \neq i}}^{N} \boldsymbol{\sigma}_k(\mathbf{x}_i) \right]$$

where $\mathbf{x}_i$ and $\mathbf{n}_i$ denote the position vector along the ith microcrack and the unit normal vector to it; $\hat{\boldsymbol{\sigma}}(\mathbf{x}_i)$ and $\boldsymbol{\sigma}_k(\mathbf{x}_i)$ are the actual stresses generated by the main crack and the kth microcrack along the line l_i of the ith microcrack.

Denoting the CODs of the main crack and of the microcracks by vectorial functions $\mathbf{b}_0(x_0)$ and $\mathbf{b}_i(x_i)$, correspondingly (x_0 and x_i are the coordinates along the main crack and ith microcrack) we can represent the stress fields $\hat{\boldsymbol{\sigma}}$ and $\boldsymbol{\sigma}_i$ as $\mathbf{T}_x \int_{l_0} \mathbf{b}_0(x') \cdot \boldsymbol{\Phi}(x', \mathbf{x})\, \mathrm{d}x'$ and $\mathbf{T}_x \int_{l_i} \mathbf{b}_i(x') \cdot \boldsymbol{\Phi}(x', \mathbf{x})\, \mathrm{d}x_i$. The $N+1$ unknown functions b_i are to be determined from $N+1$ vectorial integral equations expressing the traction-free boundary conditions on the crack faces:

(a) on microcracks:

$$\mathbf{n}_i \cdot \left[T_x \int_{l_0} \mathbf{b}_0(x') \cdot \boldsymbol{\Phi}(x', x_i)\, \mathrm{d}x' + \sum_{\substack{k=1 \\ k \neq i}}^{N} \mathbf{T}_x \int_{l_k} \mathbf{b}_k(\mathbf{x}') \cdot \boldsymbol{\Phi}(\mathbf{x}', \mathbf{x}_i)\, \mathrm{d}x' + \mathbf{T}_x \int_{l_i} \mathbf{b}_i(\mathbf{x}') \cdot \boldsymbol{\Phi}(\mathbf{x}', \mathbf{x}_i)\, \mathrm{d}x' + \boldsymbol{\sigma}^{\infty} \right] = 0 \quad (6)$$

$$i = 1, \ldots, N$$

(It is convenient, for computational purposes, to move the stress operator $\mathbf{T}_x$ under the integral sign. Then the last integral in the brackets becomes divergent, even in the principal value sense, so that direct substitution $x = x_i$ in the boundary condition on the ith crack becomes impossible. It can be shown that the limiting value of this integral as $x \to x_i$ is given by the following regularization[8]

$$\lim_{x \to x_i} \mathbf{T}_x \int_{l_i} \mathbf{b}(x') \cdot \boldsymbol{\Phi}(\mathbf{x}', \mathbf{x})\, \mathrm{d}x' = \int_{l_i} [\mathbf{b}(\mathbf{x}') - b(\mathbf{x})] \cdot \mathbf{T}_x \{ \boldsymbol{\Phi}(\mathbf{x}', \mathbf{x}) \}\, \mathrm{d}x'$$

where the integral on the right is to be understood in the principal value sense.)

(b) on the main crack:

$$\mathbf{n}_0 \cdot \left[\sum_{i=1}^{N} T_x \int_{l_i} \mathbf{b}_i(x') \cdot \boldsymbol{\Phi}(x', x_0)\, \mathrm{d}x' + \mathbf{T}_x \int_{l_0} \mathbf{b}_0(x') \cdot \boldsymbol{\Phi}(x', x_0)\, \mathrm{d}x' + \boldsymbol{\sigma}^{\infty} \right] = 0 \quad (7)$$

and the same remark on regularization of the last integral in the brackets applies.

GENERAL FORM OF THE SOLUTION

The system of integral equations, eqns (6) and (7), can be conveniently reduced to a system of linear algebraic equations by making use of Willis' polynomial conservation theorem[6]. Willis' results are further advanced by deriving the actual relations between the coefficients of both polynomials. Using these relations and approximating the tractions induced along a given microcrack line by Taylor's polynomials centered at the microcrack center we reduce the problem to a system of linear algebraic equations for the polynomial's coefficients. Note that the linear polynomial approximation, $P = 1$, i.e. representation of the microcrack CODs as "linearly distorted" ellipses may often be sufficient. If the traction induced on each of the microcracks l_i by the main crack field is dominant compared to the tractions due to other microcracks, then, since the microcracks are small, the main crack field does not change much along each of them and the *piecewise* *constant approximation* ($P = 0$) appears to be adequate. The examples considered in Part II can serve as an illustration.

Thus, the boundary conditions on the microcracks can be rewritten in the form

$$-\mathbf{n}_i\cdot\mathbf{T}_x\int_{l_i}\mathbf{b}_i(x')\cdot\mathbf{\Phi}(x',\mathbf{x}_i)\,\mathrm{d}x' = \mathbf{n}_i\cdot\left[\hat{\sigma}(x_i)+\sum_{\substack{k=1\\k\neq i}}^{N}\mathbf{T}_x\int_{l_k}\mathbf{b}_k(x')\cdot\mathbf{\Phi}(x',\mathbf{x}_i)\,\mathrm{d}x'+\boldsymbol{\sigma}^{\infty}\right]. \quad (6a)$$

The left-hand side represents the traction on the ith microcrack, the right-hand side represents the sum of the tractions induced on l_i by the rest of the microcracks by the main crack and by the remote loading. The terms on the right-hand side are analytic along l_i and can be represented there by Taylor's polynomials centered at the ith microcrack center $\mathbf{x}_i^0$:

$$\hat{\sigma}_0(\mathbf{x}_i) \simeq \sum_{m=1}^{P}\hat{\sigma}^{(m)}(\mathbf{x}_i^0)\frac{(\mathbf{x}_i-\mathbf{x}_i^0)^m}{m!};\qquad \boldsymbol{\sigma}_k(\mathbf{x}_i) \simeq \sum_{m=1}^{P}\boldsymbol{\sigma}_k^{(m)}(\mathbf{x}_i^0)\frac{(\mathbf{x}_i-\mathbf{x}_i^0)^m}{m!}. \quad (8)$$

Then the overall traction on the ith microcrack becomes a polynomial of degree P and, according to the polynomial conservation theorem, the COD of the ith microcrack is a "polynomially distorted" ellipse, with a Pth degree polynomial:

$$\mathbf{b}_i(\mathbf{x}_i) = \sum_{m=0}^{P}\mathbf{b}_i^{(m)}(\mathbf{x}_i^0)\frac{(\mathbf{x}_i-\mathbf{x}_i^0)^m}{m!}\,e_i(\mathbf{x}_i) \quad (9)$$

where $e_i(\mathbf{x}_i)$ denotes an arc of the ellipse of unit opening with the tips coinciding with the ith crack tips. Substituting the polynomial representations, eqns (8) and (9), into (6a) and equating the polynomial's coefficients we obtain a system of $2NP$ linear algebraic equations for $2NP$ components of $\mathbf{b}_i^{(m)}(\mathbf{x}_i^0)$. These relations also contain the unknown quantities $\hat{\sigma}^{(m)}$—the derivatives of the main crack field $\hat{\sigma}(\mathbf{x}) = \mathbf{T}_x\int_{l_0}\mathbf{b}_0(x')\cdot\mathbf{\Phi}(x',\mathbf{x})\,\mathrm{d}x'$ with the COD $\mathbf{b}_0$ of the main crack being an unknown function. The additional equations are provided by the boundary condition on the main crack. It can be rewritten in the form similar to eqn (6a), with the left-hand side of the equation being the traction on the main crack and the right-hand side being the sum of the tractions induced on l_0 by the microcracks and by the remote field $\boldsymbol{\sigma}^{\infty}$:

$$-\mathbf{n}_0\cdot\mathbf{T}_x\int_{l_0}\mathbf{b}_0(x')\cdot\mathbf{\Phi}(x',x_0)\,\mathrm{d}x' = \mathbf{n}_0\cdot\left[\sum_{i=1}^{N}\mathbf{T}_x\int_{l_i}\mathbf{b}_i(x')\cdot\mathbf{\Phi}(x',x_0)\,\mathrm{d}x'+\boldsymbol{\sigma}^{\infty}\right]. \quad (7a)$$

Using the polynomial representations (9) for $\mathbf{b}_i$, approximating the tractions along l_0 given by the right-hand side of eqn (7a) by Taylor's polynomials centered at the main crack center $\mathbf{x}_0^0$, representing the COD $\mathbf{b}_0$ of the main crack in the form (ellipse) × (polynomial) and, finally, invoking the relations between the coefficients of polynomial loading and the polynomial coefficients of COD, we obtain the additional $2P$ relations for the components of $\mathbf{b}_i^{(m)}(x_i^0)$ and $\mathbf{b}_0^{(m)}(x_0^0)$.

The only known quantities contained in the mentioned relations are $\mathbf{n}_k\cdot\boldsymbol{\sigma}^{\infty(m)}(\mathbf{x}_k^0)$ ($k = 0, 1, \ldots, N$)-directional derivatives of the remote field taken along the crack lines and evaluated at their centers $\mathbf{x}_k^0$ or, in the case of a uniform field $\boldsymbol{\sigma}^{\infty}$, the tractions $\mathbf{n}_k\cdot\boldsymbol{\sigma}^{\infty}$. The unknowns $-\mathbf{b}_i$'s and $\mathbf{b}_0$ can therefore be found in terms of the $\boldsymbol{\sigma}^{\infty}$-related quantities; the total stress field in the microcracking zone can then be determined by substitution of the CODs in the potential representation (3).

Below, the important case of "*small scale*" *microcracking* is considered when the microcracks can be treated as embedded into the crack tip field of the main crack. Then the stress $\boldsymbol{\sigma}^{\infty}$ can be neglected as compared to $\hat{\sigma}$. We assume, for simplicity of calculations, that the main crack undergoes a pure mode I loading and that the microcrack array is symmetric with respect to the main crack line, so that the mode I conditions on the main crack are not violated. Then $\hat{\sigma}(\mathbf{x}) = K_I^{\text{eff}}\sigma_0(\mathbf{x})$ where $\sigma_0(\mathbf{x}) = \varphi[\theta(\mathbf{x})]/\sqrt{(2\pi r(\mathbf{x}))}$ is the standard mode I crack tip field (r and θ are polar coordinates in the main crack tip coordinate system) and K_I^{eff} is the "effective" stress intensity factor at the main crack tip, with the account of the influence of microcracks.

Thus, the stress field generated by the main crack, can be expressed in terms of just one unknown parameter K_I^{eff} (rather than a number of unknown coefficients $\mathbf{b}_0^{(m)}$ of Taylor's approximation of $\mathbf{b}_0$). Omitting σ^∞, we can represent the stress field in the microcracking zone in the form

$$\sigma(\mathbf{x}) = K_I^{\text{eff}}\sigma_0(\mathbf{x}) + \sum_{i=1}^{N} \mathbf{T}_x \int_{l_i} \mathbf{b}_i(x_i') \cdot \mathbf{\Phi}(x_i', \mathbf{x})\, \mathrm{d}x_i'. \tag{10}$$

An additional equation for K_I^{eff} reflects the impact of the microcrack array:

$$K_I^{\text{eff}} = K_I^0 + \frac{1}{\sqrt{(\pi l_0)}} \int_{-l_0}^{l_0} \sqrt{\left(\frac{l_0+\xi}{l_0-\xi}\right)}\mathbf{n} \cdot \sum_{i=1}^{N} \sigma_i(x) \cdot \mathbf{n}\, \mathrm{d}x \tag{11}$$

(K_I^0 is the stress intensity factor in the absense of microcracks). Expressions (10) and (11) represent stresses and effective stress intensity factor in a close vicinity of the main crack tip. The only known quantities contained in this system are the field $K_I^0\sigma_0$ and its directional derivatives $\mathbf{n}_i \cdot K_I^0\sigma_0^{(m)}(x_i^0)$ evaluated at the microcrack centers. The unknowns—microcrack CODs can therefore be expressed in terms of these quantities; the stress field in the microcracking zone can then be found by substitution of the CODs in the potential representation (10).

It should be noted that substitution of the boundary condition (7a) on the main crack by relation (11) for K_I^{eff}, being an adequate representation of K_I^{eff}, is not a fully adequate replacement in general; namely, the traction-free boundary conditions on the main crack may not be satisfied. This means that the stress field $\sum \sigma_i$ generated by the microcracks may result in non-zero tractions on the main crack but these tractions will produce a zero contribution to the stress intensity factor K_I^{eff} and thus will not affect the analysis in the microcracking zone. Hence, in the vicinity of the main crack tip, the following general statement on the structure of the stress field in the damage zone can be formulated:

> The stress field in the "small scale" microcracking zone is a linear combination of the mode I main crack tip field $K_I^0\sigma_0(\mathbf{x})$ in the absence of microcracks and the stress fields generated by microcracks. The latter are loaded by polynomial tractions which are linear combinations of the directional derivatives $K_I^0\mathbf{n}_i \cdot \sigma_0^{(m)}(\mathbf{x}_i^0)$ taken along the microcracks and evaluated at their centers. Taking higher order polynomial approximations, i.e. including higher order derivatives of σ_0, we can make the solution as close to the exact stress field as desired.

In the piecewise constant approximation, the mentioned tractions are combinations of projections of the $K_I\sigma_0$-field onto the microcrack lines.

To make this statement in a more constructive form, the following notations are introduced:

(1) $\{\mathbf{t}\}$ is the set of mth directional derivatives $\mathbf{t}^{(m)}(\mathbf{x}_s^0) = \mathbf{n}_s \cdot K_I^0\sigma_0^{(m)}(\mathbf{x}_s^0)$ of the main crack tip field (in the absense of microcracks) in the direction of the sth microcrack evaluated at its center $\mathbf{x}_s^0$. Note that the directional derivatives of higher orders are understood in the following sense: if $y_s - y_s^0 = \alpha(x_s - x_s^0)$ is the equation of the sth microcrack line so that the stress $\sigma_0 = \sigma_0[x_s, y_s^0 + \alpha(x_s - x_s^0)]$ is a function of x_s only, then $\sigma_0^{(m)}(\mathbf{x}_s^0) = \mathrm{d}^m/\mathrm{d}x_s^m\sigma_0[x_s, y_s^0 + \alpha(x_s - x_s^0)]$.

(2) $\{\mathbf{b}\}$ is the set of vector coefficients $\mathbf{b}_k^{(n)}$ of a polynomial which is superimposed on the elliptical COD $e(\xi_k)$ of the kth microcrack.

(3) $\{\mathbf{B}\} = B_{mn}(\mathbf{x}_s^0, \mathbf{x}_k^0)$ is a linear operator which depends on the positions of the centers of

two microcracks $\mathbf{x}_s^0, \mathbf{x}_k^0$ and characterizes the influence of the mth derivative of the sth microcrack stress field on the nth derivative of coefficient $\{\mathbf{b}\}$ of the kth microcrack COD. The expression for $\{\mathbf{B}\}$ is given below.

(4)
$$\{\mathbf{A}(\mathbf{x})\} = \mathbf{A}_n(\mathbf{x}^k, x) = \mathbf{T}_x \int_{\omega_k} \frac{(\xi_k - \xi_k^0)^n}{n!} e(\xi_k)\mathbf{\Phi}(\xi_k, \mathbf{x})\, \mathrm{d}\xi_k \tag{12}$$

is a matrix which characterizes the stress associated with the nth derivative of the microcrack opening coefficient $\{\mathbf{b}\}$ of the kth microcrack at arbitrary point $\mathbf{x}$. Matrix $\mathbf{A}(\mathbf{x})$ is the matrix of the microcrack array; ξ^k is a coordinate on the kth microcrack, and the integral is taken along the kth microcrack of length $2l_k$. The components of matrix $\mathbf{A}(\mathbf{x})$ are the third rank tensors.

(5) $\{\mathbf{A}^0\} = \mathbf{A}_n^0(\mathbf{x}^k)$ is a linear operator which characterizes the increase of stress intensity factor K_I^0 due to stresses associated with microcrack opening coefficients $\{\mathbf{b}\}$ of the array. The components of linear operator $\{\mathbf{A}^0\}$ are vectors. The definition of $\{\mathbf{A}^0\}$ is given below.

The unknown stress field $\boldsymbol{\sigma}(\mathbf{x})$, in the approximation of a "small scale" microcracking, can be expressed in terms of the asymptotic stress field of the main crack $\boldsymbol{\sigma}_0(\mathbf{x})$, the values of its derivatives $\sigma_0^{(m)}(x_0)$ in the directions of microcracks evaluated at the centers of microcracks, and the second Green's tensor $\mathbf{\Phi}(\xi, \mathbf{x})$ by the formula

$$\boldsymbol{\sigma}(\mathbf{x}) = (\{\mathbf{I}\} + \{\mathbf{B}\}\{\mathbf{t}\}\{\mathbf{A}^0\})\boldsymbol{\sigma}_0(\mathbf{x}) + \{\mathbf{B}\}\{\mathbf{t}\}\{\mathbf{A}(\mathbf{x})\} \tag{13}$$

or, in index notation

$$\boldsymbol{\sigma}(\mathbf{x}) = \left[\mathbf{I} + \sum_{m,n,s,k} \mathbf{B}_{mn}(\mathbf{x}_0^s, \mathbf{x}_0^k)\mathbf{t}_m(\mathbf{x}^s)\mathbf{A}^0(\mathbf{x}^k)\right]\boldsymbol{\sigma}_0(\mathbf{x})$$
$$+ \sum_{m,n,s,k} \mathbf{B}_{mn}(\mathbf{x}_0^s, \mathbf{x}_0^k)\mathbf{t}_m(\mathbf{x}_0^s)\mathbf{A}_n(\mathbf{x}^k, x) \tag{13a}$$

where $\{\mathbf{I}\}$ is a unit operator.

The last formula can be obtained by the following line of reasoning:

The stress field $\boldsymbol{\sigma}(\mathbf{x})$ for the small scale model is given by formula (6). Equilibrium equations are automatically satisfied for (6) because of the properties of the asymptotic crack tip solution and the properties of the second Green's tensor $\mathbf{\Phi}(\mathbf{x}', \mathbf{x})$.

The equations to be satisfied are the boundary conditions on the macrocrack and each of the microcracks. For the small scale model, the boundary condition on the macrocrack can be substituted by the equation for effective stress intensity factor $K_\mathrm{I}^{\mathrm{eff}}$ which fully determines the asymptotic stress field of the macrocrack

$$K_\mathrm{I}^{\mathrm{eff}} = K_\mathrm{I}^0 + \frac{1}{\sqrt{(\pi l_0)}} \int_{-l_0}^{l_0} \sqrt{\left(\frac{l_0 + x}{l_0 - x}\right)} \mathbf{n}(\mathbf{x})\mathbf{n}(\mathbf{x}) \cdot \sum_{i=1}^{N} \boldsymbol{\sigma}_i(\mathbf{x})\, \mathrm{d}x. \tag{14}$$

Boundary conditions on the microcracks form a system of $2N$ singular integral equations for $2N$ unknown components of N vectors of double layer potential densities $\mathbf{b}(\mathbf{x})$ $(K = 1, 2, \ldots, N)$.

$$\mathbf{n}_i\left\{\hat{\boldsymbol{\sigma}}(\mathbf{x}_i) + \sum_{\substack{k=1\\k\neq i}}^{N} \mathbf{T}_x \int_{l_k} \mathbf{b}_k(\mathbf{x}') \cdot \mathbf{\Phi}(\mathbf{x}', \mathbf{x}_i)\, \mathrm{d}x' + \mathbf{T}_x \int_{l_i} \mathbf{b}_i(\mathbf{x}') \cdot \mathbf{\Phi}(\mathbf{x}', \mathbf{x}_i)\, \mathrm{d}x'\right\} = 0 \tag{15}$$

where $\mathbf{x}_i$ is a coordinate on the ith microcrack. Equations (14) and (15) constitute a system for determination of $2N + 1$ unknowns $K_\mathrm{I}^{\mathrm{eff}}$ and the components of N vectors $\mathbf{b}_k(\mathbf{x}')$.

Using (11) and (12) the boundary conditions may be written as follows:

$$\mathbf{n}(\mathbf{x}^k)\cdot\mathbf{T}_x\int_{l_k}\sum_{n=0}^{P}\mathbf{b}^{(n)}(\xi_0^k)\frac{(\xi^k-\xi_0^k)^n}{n!}e(\xi^k)\mathbf{\Phi}(\xi_k,\mathbf{x}_k)\,\mathrm{d}\xi_k =$$

$$-\sum_{n=0}^{P}\sum_{s=1}^{N}\mathbf{n}(\mathbf{x}_0^k)\cdot\mathbf{T}_x\int_{l_s}\mathbf{b}^{(n)}(\xi_0^s)\frac{(\xi^s-\xi_0^s)^n}{n!}e(\xi^s)\mathbf{\Phi}(\xi^s,\mathbf{x}^k)\,\mathrm{d}\xi_s$$

$$-K_1^{\text{eff}}\mathbf{s}(\mathbf{x}^k) \tag{16}$$

where

$$\mathbf{s}(\mathbf{x}^k) = \mathbf{n}(\mathbf{x}^k)\frac{\varphi[\theta(\mathbf{x})]}{\sqrt{(2\pi r(x))}}. \tag{17}$$

The expression on the left of (16) represents traction on the Kth microcrack which, by assumption, may be represented as a polynomial. (The left-hand side of (16) contains a singular integral, when $n = 0$, which converges in Cauchy's sense.) Consequently, the right-hand side of (16) (i.e. the traction on the Kth microcrack induced by the rest of the microcracks and the dominant field) is a polynomial also; then (16) can be rewritten in the form of a system of equations for the coefficients of the polynomials of the left-hand and right-hand parts of (16). The procedure described above can be carried out by differentiation of (16) at point $\mathbf{x}_0^k$ in the direction of the kth microcrack P times. The differentiation results in the following equation:

$$\sum_{n=0}^{P}\sum_{k=1}^{N}\mathbf{I}_{mn}(\mathbf{x}_0^s,\mathbf{x}_0^k)\mathbf{b}_k^n = -\sum_{n=0}^{P}\sum_{k=1}^{N}\mathbf{H}_{mn}(\mathbf{x}_0^s,\mathbf{x}_0^k)\mathbf{b}_k^n - K_1^{\text{eff}}\mathbf{S}_m(\mathbf{x}_0^s) \tag{18}$$

where $\mathbf{S}_m(\mathbf{x}_0^s)$ is the mth derivative on the sth microcrack, $m = 0,1,2,\ldots,P, S = 1,2,\ldots,N$, $\mathbf{b}_k^n = \mathbf{b}^{(n)}(\mathbf{x}_0^k)$. The linear operator $\mathbf{H}_{mm}(\mathbf{x}_0^s\mathbf{x}_0^k)$ is given by the following expression:

$$\mathbf{H}_{mn}(x_0^s,x_0^k) = \begin{cases} 0\ \ldots\ldots\ \text{when } s = k \\ \mathbf{T}_x\cdot\displaystyle\int_s\frac{(\xi^s-\xi_0^s)^n}{n!}e(\xi^s)\mathbf{\Phi}^{(m)}(\xi^s,\mathbf{x}_0^k)\cdot\mathbf{n}(\mathbf{x}_0^k)\,\mathrm{d}\xi^s \\ \qquad\text{when } s \neq k \end{cases} \tag{19}$$

taken at point $\mathbf{x} = \mathbf{x}_0^k$ where $\mathbf{\Phi}^{(m)}(\xi^s,\mathbf{x}_0^k)$ is the mth directional derivative of Green's tensor at point k taken in the direction of the kth microcrack. Similarly, the linear operator $I_{mm}(x_0^s,x_0^k)$ is given by the expression:

$$I_{mn}(\mathbf{x}_0^s\mathbf{x}_0^k) = \begin{cases} 0\ \ldots\ldots\ \text{when } s \neq k \\ \mathbf{T}_x\cdot\displaystyle\int_{l_s}\frac{(\xi^s-\xi_0^s)^n}{n!}e(\xi^s)\mathbf{\Phi}^{(m)}(\xi^s,\mathbf{x}_0^k)\mathbf{n}(\mathbf{x}_0^k)\,\mathrm{d}\xi^s \\ \qquad\text{when } s = k. \end{cases} \tag{20}$$

The set of elements $\mathbf{b}_k^n$ in (18) constitutes a $(P+1)\times N$ matrix. In symbolic notation equation (18) may be written as follows:

$$\{\mathbf{I}\}\cdot\{\mathbf{b}\} = -\{\mathbf{H}\}\cdot\{\mathbf{b}\} - K_1^{\text{eff}}\cdot\{\mathbf{s}\} \tag{18a}$$

together with eqn (14) for the stress intensity factor K_1^{eff} boundary conditions on microcracks (15) form a system of $(P+1)\times 2N+1$ linear algebraic equations with $(P+1)\times 2N$ unknown components of matrix $\{\mathbf{b}\}$ and unknown quantity K_1^{eff}. Formula (14) with $\boldsymbol{\sigma}_i(x)$ given by (3) and $\mathbf{b}_i(x)$ given by (12) takes the form

$$K_1^{\text{eff}} = K_1^0 + \{\mathbf{A}^0\}\cdot\{\mathbf{b}\} \tag{21}$$

where linear operator $\{\mathbf{A}^0\}$ is given by the expression

$$\{\mathbf{A}^0\} = \frac{1}{\sqrt{(\pi l_0)}}\int_{-l_0}^{l_0}\sqrt{\left(\frac{l_0+x}{l_0-x}\right)}\mathbf{n}(x)\sum_{n=1}^{P}\sum_{s=0}^{N}\mathbf{H}_{0n}(\mathbf{x}_0^s,\mathbf{x})\,\mathrm{d}x \tag{22}$$

with $\mathbf{H}_{0n}(\mathbf{x}_0^s,\mathbf{x})$ defined by (19). Formula (22) defines linear operator $\{\mathbf{A}^0\}$. Operator $\{\mathbf{A}^0\}$ characterizes the impact of a microcrack array on the main crack.

Substitution of (21) into (18a) gives a system of linear algebraic equations for the determination of $\{\mathbf{b}\}$

$$\{\mathbf{I}\}\{\mathbf{b}\} = -(\{\mathbf{H}\}+\{\mathbf{s}\}\{\mathbf{A}^0\})\{\mathbf{b}\}-\{\mathbf{t}\}. \tag{23}$$

The last system of equations yields an obvious solution

$$\{\mathbf{b}\} = \{\mathbf{B}\}\{\mathbf{t}\} \tag{24}$$

where

$$\{\mathbf{B}\} = -(\{\mathbf{I}\}+\{\mathbf{H}\}+\{\mathbf{s}\}\{\mathbf{A}^0\})^{-1}. \tag{25}$$

Formula (25) defines the linear operator $\{\mathbf{B}\}$.

In index notation (24) takes the form

$$\mathbf{b}_k^n = \sum_{s=1}^{N}\sum_{m=0}^{P}\mathbf{B}_{mn}(\mathbf{x}_0^s,\mathbf{x}_0^k)\mathbf{t}_m(\mathbf{x}_0^s). \tag{24a}$$

Formula (24) represents $\{\mathbf{b}\}$ as a linear function of directional derivatives of the asymptotic stress field $\boldsymbol{\sigma}_0(\mathbf{x})(\mathbf{n}\cdot\boldsymbol{\sigma}_0(\mathbf{x}) = K_I^0\mathbf{s}(\mathbf{x}))$. Substitution of (24) and (21) into (6) furnishes (13).

Thus, taking into account the analytic character of the solution of a plane problem of elastostatics, the resulting stress field $\boldsymbol{\sigma}(\mathbf{x})$ can be approximated by a polynomial as closely as desired. Consequently, the solution obtained above can be made as close as desired to the exact one.

It should be noted that each term in formula (13) for the resulting stress field $\boldsymbol{\sigma}(\mathbf{x})$ can be given a clear physical interpretation; the first term represents the dominating stress field of the main crack $\hat{\boldsymbol{\sigma}}(\mathbf{x})$ and the second term represents the stress field of the microcrack array imbedded into the stress field of the macrocrack $\boldsymbol{\sigma}(\mathbf{x})$. Dominating stress field $\hat{\boldsymbol{\sigma}}(\mathbf{x})$, in turn, consists of two terms, the first being the asymptotic stress field of the main crack $\boldsymbol{\sigma}_0(\mathbf{x})$, and the second term results from the impact of the microcrack array (embedded into asymptotic stress field) on the macrocrack.

ITERATIVE APPROACH

If the number of microcracks N is large then the system of $2N(P+1)+1$ scalar linear algebraic equations may become large and inconvenient for analytical solution, even in a piecewise constant approximation $P = 0$. This is particularly so if the microcrack array geometry is known only in probabilistic terms.

An iterative approach can be suggested as an alternative to a direct solution. In formulas (10, 11)

$$\boldsymbol{\sigma}(\mathbf{x}) = K_I^{\mathrm{eff}}\boldsymbol{\sigma}_0(\mathbf{x})+\sum_{i=1}^{N}\mathbf{T}_x\int_{l_i}\mathbf{b}_i(x_i')\cdot\boldsymbol{\Phi}(x_i',\mathbf{x})\,\mathrm{d}x_i'$$

$$K_I^{\mathrm{eff}} = K_I^0+\frac{1}{\sqrt{(\pi l_0)}}\int_{-l_0}^{l_0}\sqrt{\left(\frac{l_0+\xi}{l_0-\xi}\right)}\mathbf{n}\cdot\sum_{i=1}^{N}\sigma_i(x)\cdot\mathbf{n}\,\mathrm{d}x$$

assume, as a zeroth approximation, $\boldsymbol{\sigma} = K_I^0\boldsymbol{\sigma}_0$ (main crack tip stress field in the absence of microcracks). Successive iterations are obtained by introducing $\boldsymbol{\sigma}$ as given by (10) back into (10, 11). The first iteration yields

$$\boldsymbol{\sigma} = K_I^{\text{eff}}\boldsymbol{\sigma}_0+\sum \boldsymbol{\sigma}_i(K_I^0\boldsymbol{\sigma}_0) \tag{26}$$

where $\boldsymbol{\sigma}_i(K_I^0\boldsymbol{\sigma}_0)$ denotes the stress field generated by the ith microcrack embedded into the stress field $K_I^0\boldsymbol{\sigma}_0$. Also,

$$K_I^{\text{eff}} = K_I^0+\frac{1}{\sqrt{(\pi l_0)}}\int_{-l_0}^{l_0}\sqrt{\left(\frac{l_0+\xi}{l_0-\xi}\right)}\mathbf{nn}:\sum_i \boldsymbol{\sigma}_i(K_I^0\boldsymbol{\sigma}_0)\,\mathrm{d}\xi \equiv K_I^0(1+q) \tag{27}$$

so that

$$\boldsymbol{\sigma} = (1+q)K_I^0\boldsymbol{\sigma}_0+\sum \boldsymbol{\sigma}_i(K_I^0\boldsymbol{\sigma}_0). \tag{28}$$

Thus, the stress field, in the first iteration, is the sum of the following fields: (1) $K_I^0\boldsymbol{\sigma}_0$; (2) $qK_I^0\boldsymbol{\sigma}_0$, i.e. correction to the main crack tip field due to the impact of the microcrack array, with the latter being embedded into the $K_I^0\boldsymbol{\sigma}_0$-field; (3) $\sum_i \boldsymbol{\sigma}_i(K_I^0\boldsymbol{\sigma}_0)$-response of the microcrack array to the $K_I\boldsymbol{\sigma}_0$-field.

The second iteration yields

$$\boldsymbol{\sigma} = K_I^{\text{eff}}\boldsymbol{\sigma}_0+\sum_i \boldsymbol{\sigma}_i(K_I^{\text{eff}}\boldsymbol{\sigma}_0)+\sum_i\sum_k \boldsymbol{\sigma}_i[\boldsymbol{\sigma}_k(K_I^0\boldsymbol{\sigma}_0)]$$

where

$$\begin{aligned}K_I^{\text{eff}} &= K_I^0+\frac{1}{\sqrt{(\pi l_0)}}\int_{-l_0}^{l_0}\sqrt{\left(\frac{l_0+\xi}{l_0-\xi}\right)}\mathbf{nn}:\sum_i \boldsymbol{\sigma}_i\left[(1+q)K_I^0\boldsymbol{\sigma}_0+\sum_k \boldsymbol{\sigma}_k(K_I^0\boldsymbol{\sigma}_0)\right]\mathrm{d}\xi \\ &= K_I^0(1+q+q^2+q^3)\end{aligned}$$

and

$$q^3 \equiv \frac{1}{\sqrt{(\pi l_0)}}\int_{-l_0}^{l_0}\sqrt{\left(\frac{l_0+\xi}{l_0-\xi}\right)}\mathbf{nn}:\sum_i\sum_k \boldsymbol{\sigma}_i[\boldsymbol{\sigma}_k(K_I^0\boldsymbol{\sigma}_0)](\xi)\,\mathrm{d}\boldsymbol{\xi}. \tag{29}$$

The new terms in the expression for $\boldsymbol{\sigma}$ introduced in the second iteration are: (1) $(q^2+q^3)K_I^0\boldsymbol{\sigma}_0$—next correction to the main crack tip field; (2) $\sum \boldsymbol{\sigma}_i(K_I^{\text{eff}}-K_I^0)\boldsymbol{\sigma}_0 = (q^2+q^3)\sum \boldsymbol{\sigma}_i(K_I^0\boldsymbol{\sigma}_0)$-response of the microcrack array to the mentioned correction, and (3) $\boldsymbol{\sigma}_i[\boldsymbol{\sigma}_k(K_I^0\boldsymbol{\sigma}_0)]$-response of the ith microcrack to the stress field generated by the kth microcrack embedded into the $K_I^0\boldsymbol{\sigma}_0$-field; thus, the double sum in the formulas of the second iteration represents a "secondary" response of the microcrack array to the $K_I^0\boldsymbol{\sigma}_0$-field. Similarly, higher order iterations will represent triple, quadruple, etc. interactions. Figure 1 illustrates the crack interactions involved, up to the second iteration (stress field generated by a microcrack is marked by two zigzag arrowhead lines originating at the crack). Note that two physically different groups of terms can be distinguished; the main crack tip fields ($K_I^{\text{eff}}\boldsymbol{\sigma}_0$, with corrections to K_I^{eff} accounting for the impact of the microcrack array) and the "diffused" stress field-sum of the microcrack generated fields.

Note that the actual calculation of the responses $\boldsymbol{\sigma}_i(K_I^0\boldsymbol{\sigma}_0)$, $\boldsymbol{\sigma}_i[\boldsymbol{\sigma}_k(K_I^0\boldsymbol{\sigma}_0)]$, etc. involves the polynomial approximation of the tractions induced along l_i by the other cracks and the corresponding polynomial representation of $\mathbf{b}_i$ in the potential formula $\boldsymbol{\sigma}_i(\mathbf{x}) = \mathbf{T}_x\int_{l_i}\mathbf{b}_i(\xi)\cdot\boldsymbol{\Phi}(\xi,\mathbf{X})\,\mathrm{d}\xi$. Thus, the solution $\boldsymbol{\sigma}(\mathbf{x})$ for the stress field in the damage zone constructed by iterations assumes two different types of approximation: by the degree P of the polynomial representation of the tractions along l_i (usually a piecewise constant $P = 0$ or

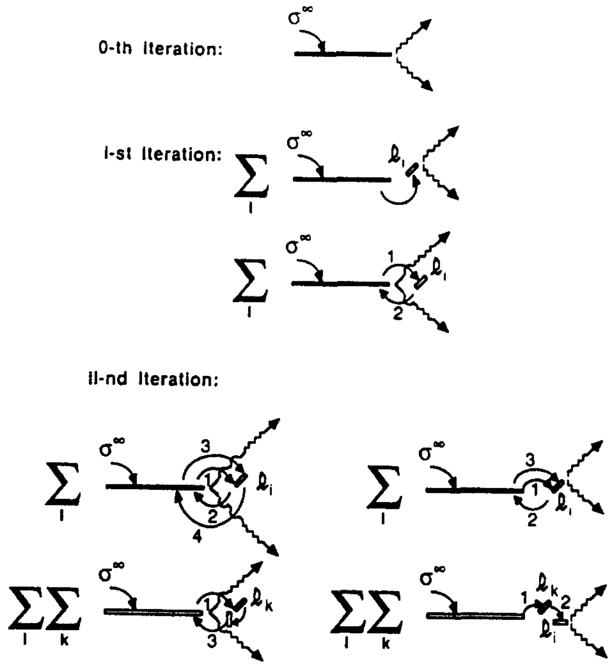

Fig. 1.

linear $P = 1$ approximation is sufficient) and by the multiplicity of crack interactions taken into account.

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